Exercise 28

Solve the boundary-value problem, if possible.

$$y'' - 8y' + 17y = 0, \quad y(0) = 3, \quad y(\pi) = 2$$

Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rx}$.

$$y = e^{rx} \quad \to \quad y' = re^{rx} \quad \to \quad y'' = r^2 e^{rx}$$

Plug these formulas into the ODE.

$$r^2 e^{rx} - 8(re^{rx}) + 17(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 - 8r + 17 = 0$$

Solve for r.

$$r = \frac{8 \pm \sqrt{64 - 4(1)(17)}}{2} = \frac{8 \pm \sqrt{-4}}{2} = 4 \pm i$$

$$r = \{4 - i, 4 + i\}$$

Two solutions to the ODE are $e^{(4-i)x}$ and $e^{(4+i)x}$. By the principle of superposition, then,

$$\begin{aligned}
y(x) &= C_1 e^{(4-i)x} + C_2 e^{(4+i)x} \\
&= C_1 e^{4x} e^{-ix} + C_2 e^{4x} e^{ix} \\
&= e^{4x}(C_1 e^{-ix} + C_2 e^{ix}) \\
&= e^{4x}[C_1(\cos x - i \sin x) + C_2(\cos x + i \sin x)] \\
&= e^{4x}[(C_1 + C_2)\cos x + (-iC_1 + iC_2)\sin x] \\
&= e^{4x}(C_3 \cos x + C_4 \sin x)
\end{aligned}$$

Apply the boundary conditions to determine C_3 and C_4.

$$y(0) = C_3 = 3$$

$$y(\pi) = -C_3 e^{4\pi} = 2$$

Solving this system of equations yields $C_3 = 3$ and $C_3 = -2e^{-4\pi}$, which is a contradiction. Therefore, there's no solution to the boundary value problem.